Seat No. : _______________

ZU-109

May-2014

M.Sc., Sem.-II

MAT-411 : Mathematics

(Real Analysis)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Attempt any **one** : **7**

   (i) State and prove Egorov's theorem.

   (ii) Let $f : E = [a, b] \to R$ be measurable. Prove that for given $\sigma > 0$ and $\in > 0$, there exists a continuous function $\psi(x)$ on $[a, b]$ such that $mE(|f - \psi| \geq \sigma) < \in$.

   (b) Attempt any **two** : **4**

   (i) Verify Egorov's theorem for the sequence $f_n : [0, 1] \to R$ defined by $f_n(x) = x^n + 1$.

   (ii) If $f_n \Rightarrow f$ and g is bounded then show that $f_n g \Rightarrow fg$.

   (iii) State (only) Luzin's theorem.

   (c) Answer in brief : **3**

   (i) True or false : If $f_n \Rightarrow f$ and $g(x) = |f(x)|$, then $f_n \Rightarrow g$.

   (ii) True or false : Uniform convergence $\Rightarrow$ convergence in measure.

   (iii) Define : Convergence in measure.

2. (a) Attempt any **one** : **7**

   (i) Prove that Lp [a, b] is complete.

   (ii) If $f : R \to R$ is continuous and 2π-periodic, then show that for every $\in > 0$ there exists a trigonometric polynomial $t(x)$ such that

   $|f(x) - t(x)| < \in$.

   (b) Attempt any **two** : **4**

   (i) If $f, g \in L_p [a, b]$, then show that $\| f + g \|_p \leq \| f \|_p + \| g \|_p$.

   (ii) If $f, g \in L_p [a, b]$, then can we say that $fg \in L_p [a, b]$? Why ?

   (iii) Show that convergence in measure need not imply convergence in norm.

(c) Answer in brief : **3**

(i) Define Bernstein polynomial.

(ii) If $f_n \to f$ in L_p [a, b] then show that $\| fn \|_p \to \| f \|_p$

(iii) True or False : $C[a, b] \subset Lp\ [a, b]$ for all $p \geq 1$.

3. (a) Attempt any **one** : **7**

(i) Show that a function $f : [a, b] \to R$ is of finite variation if and only if it is a difference of two increasing functions.

(ii) Let f be an increasing function on [a, b]. Show that its derivative $f'(x)$ is measurable and

$$\int_a^b f'(x)\, dx \leq f(b) - f(a).$$

(b) Attempt any **two** : **4**

(i) If $s(x)$ denotes the saltus function of $f : [a, b] \to R$ then show that it is increasing on [a, b].

(ii) If $f(x) = \sin x$ then show that the total variation of f on R is not finite.

(iii) Compute the derived numbers of $f(x) = |x| + x$ at $x = 0$.

(c) Answer in brief : **3**

(i) True or false : Every Lipschitz continuous function on [a, b] is of finite variation.

(ii) True or false : Every function of finite variation on [a, b] is bounded.

(iii) When do we say that a function $f : R \to R$ is of finite variation ?

4. (a) Attempt any **one** : **7**

(i) If $f : [a, b] \to R$ is absolutely continuous and $f'(x) = 0$ almost everywhere then prove that $f(x)$ is constant function.

(ii) If $\phi : [a, b] \to R$ is such that at every point of [a, b] all the derived numbers of ϕ are non-negative, then prove that ϕ is increasing.

(b) Attempt any **two** : **4**

(i) If f is summable on [a, b], then prove that

$$\phi(x) = \int_a^x f(t)\, dt$$

is absolutely continuous on [a, b].

(ii) Show that the point of continuity is Lebesgue point for a summable function.

(iii) Prove that $f(x) = xe^x$ is absolutely continuous on [0,1].

(c) Answer in brief : **3**

(i) True or false : Every absolutely continuous function on [a, b] is differentiable everywhere.

(ii) Give the definition of absolutely continuous function.

(iii) Give an example of an absolutely continuous function on [0, 1].

5. (a) Attempt any **one** : **7**

(i) Show that if $f \in L[-\pi, \pi]$ is continuous at the point $x \in (-\pi, \pi)$ then its Fourier series is cesaro summable at the point x to $f(x)$.

(ii) For $f \in L_2[-\pi, \pi]$ if $s_n(x)$ denotes the partial sum of the Fourier series of f, then show that

$\| f - T_n \|_2 \geq \| f - s_n \|_2$,

For every trigonometric polynomial T_n of degree n.

(b) Attempt any **two** : **4**

(i) Give an example of an orthonormal family in $L_2[-\pi, \pi]$.

(ii) State Parseval's identity and use it to prove that if the Fourier coefficients of an L_2 – function are all zero, then the function is zero almost everywhere.

(iii) Determine the Fourier series of the 2π periodic function $f(t) = \cos^2 t + \cos t \sin t$, $-\pi \leq t \leq \pi$.

(c) Answer in brief : **3**

(i) Define : Fejer kernel

(ii) State (only) Riemann – Lebesgue lemma.

(iii) True or false : $\Sigma \frac{\sin nx}{n}$ is a Fourier series of an L_2 function.